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Multiple positive solutions of the one-dimensional p -Laplacian

Abdelkader Lakmeche^{a,*}, Ahmed Hammoudi^b^a *Nonlinear Dynamics Group, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisbon, Portugal*^b *Département de Mathématiques, Université Djillali LIABES, BP 89, Sidi Bel Abbes, Algeria*

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Abstract

In this work we investigate the existence of positive solutions of the p -Laplacian, using the quadrature method. We prove the existence of multiple solutions of the one-dimensional p -Laplacian for $\alpha \geq 0$, and determine their exact number for $\alpha = 0$.

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1. Introduction

The p -Laplacian operator was considered in several recent works (see, for instance, [1,7,10,14,17]). It arises in the modelling of different physical and natural phenomena; non-Newtonian mechanics [11,17], nonlinear elasticity and glaciology [16,17], combustion theory [24], population biology [22], nonlinear flow laws [16,19,20], system of Monge–Kantorovich partial differential equations [15]. There exists a very large number of papers devoted to the existence of solutions of the p -Laplacian operator in which the authors used bifurcation [4,13], variational methods [2,4,6,20], sub–super solutions [26], numerical methods [16], degree theory [15] or quadrature method [1,7], in order to prove the existence of solutions of this nonlinear operator.

* Corresponding author.

E-mail address: lakmeche@yahoo.fr (A. Lakmeche).

In this work we consider the following boundary value problem involving the p -Laplacian:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(u), \\ u(0) = 0, \quad u(1) + \alpha u'(1) = 0, \end{cases} \quad (1.1)$$

where $p \in (1, 2]$, $\alpha \geq 0$, $\lambda \geq 0$ and $f: R_+ \rightarrow R_+^*$ smooth enough.

We use the quadrature method to prove the existence of multiple positive solutions, this method is simple and constructive. It is usually used in the case of the linear operator ($p = 2$) in order to prove the existence of positive solutions (see, for instance, [3,5,8,9,12,18,21,25–27]).

The problem (1.1) was studied in [7] with Dirichlet boundary conditions ($\alpha = 0$). The authors proved the existence of multiple positive solutions for $\lambda \in (0, \lambda^*)$, with $\lambda^* > 0$, when f is bounded below by a positive constant and $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = +\infty$.

In this work we prove the existence of multiple positive solutions of (1.1) for $\alpha \geq 0$, and we determine their exact number in the particular case $\alpha = 0$.

Note that, for $p = 2$, the problem (1.1) is equivalent to the problem studied by Anuradha et al. [3], and for $\alpha = 0$ the problem (1.1) is equivalent to the problem studied by Bouguima and Lakmeche [7]. So, our work generalizes in a certain sense those of [3] and [7] for $\alpha \geq 0$ and $p \in (1, 2]$.

Preliminaries and definitions are given in Section 2. We give our main results in Section 3, and concluding remarks in the last section.

2. Preliminaries

In this section we give some preliminaries and definitions.

Definition 2.1. A pair $(u, \lambda) \in C^1([0, 1]; R_+) \times [0, \infty)$ is called a solution of the problem (1.1) if

- (1) $(|u'|^{p-2}u')$ is absolutely continuous, and
- (2) $-(|u'|^{p-2}u')' = \lambda f(u)$ a.e. on $(0, 1)$ and $u(0) = u(1) + \alpha u'(1) = 0$.

Remark 2.2. The pair $(0, 0)$ is a solution of (1.1).

To prove our main results, we need the following lemmas.

Let $F: R_+ \rightarrow R_+$ be defined by $F(u) := \int_0^u f(s) ds$, $g: R_+^* \rightarrow R_+^*$ be defined by

$$g(\rho) := 2 \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \int_0^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}}$$

and $h_m: [\alpha m, +\infty) \rightarrow R_+^*$ ($m > 0$, $\alpha > 0$), be defined by

$$h_m(\rho) := \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \int_0^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} + \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \int_{\alpha m}^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}}.$$

Then we have

Lemma 2.3. The functions g and h_m are continuous, $g(\rho) \leq 2h_m(\rho) \leq 2g(\rho)$, $\forall \rho > \alpha m$, and $\lim_{\rho \rightarrow 0^+} g(\rho) = 0$.

Proof. See [7, Theorem 7]. \square

For $u \in C^1([0, 1]; \mathbb{R}_+)$, we define $\|u\| := \sup\{u(s); s \in (0, 1)\}$.
Then we have

Lemma 2.4. *If (u, λ) is a solution of (1.1) with $\lambda > 0$, then*

- (1) $u'(1) < 0$,
- (2) for $\alpha > 0$, $\lambda^{1/p} = h_m(\|u\|)$, where $m = -u'(1)$, and
- (3) for $\alpha = 0$, $\lambda^{1/p} = g(\|u\|)$.

Proof. Let (u, λ) be a solution of (1.1), if $\lambda > 0$, then $u \neq 0$. Using the maximum principle (see [28]), we obtain $u > 0$ in $(0, 1)$. Since $f(0) > 0$, we have $u'(0) > 0$. Hence $u'(1) = -\frac{u(1)}{\alpha} < 0$. Then there exists a unique x_0 such that $u'(x_0) = 0$.

Moreover, $u(x_0) = \|u\|$, $u'(x) > 0$ for $x \in (0, x_0)$, and $u'(x) < 0$ for $x \in (x_0, 1)$.

Let $\rho = \|u\|$ and integrate the first equation of (1.1).

Then, for $x \in (0, x_0)$, we obtain

$$\left(\frac{p-1}{p}\right)(u'(x))^p = \lambda(F(\rho) - F(u(x))). \quad (2.1)$$

Integrating (2.1), we obtain

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{\rho} \frac{ds}{(F(\rho) - F(s))^{\frac{1}{p}}} = \lambda^{\frac{1}{p}} x_0 \quad (2.2)$$

and by symmetry we have

$$\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\alpha m}^{\rho} \frac{ds}{(F(\rho) - F(s))^{\frac{1}{p}}} = \lambda^{\frac{1}{p}} (1 - x_0), \quad (2.3)$$

where $m = -u'(1)$.

From (2.2) and (2.3) we deduce the results of Lemma 2.4. \square

Let $\rho > 0$ and $\lambda > 0$. Then, after integrating by parts, we obtain

$$g(\rho) = 2q \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \left\{ \frac{F(\rho)^{\frac{1}{q}}}{f(0)} - \int_0^{\rho} (F(\rho) - F(s))^{\frac{1}{q}} \frac{f'(s)}{f^2(s)} ds \right\}, \quad (2.4)$$

where $p^{-1} + q^{-1} = 1$.

Then

$$g'(\rho) = 2 \left(\frac{p-1}{p}\right)^{\frac{1}{p}} f(\rho) \left\{ \frac{F(\rho)^{-\frac{1}{p}}}{f(0)} - \int_0^{\rho} (F(\rho) - F(s))^{-\frac{1}{p}} \frac{f'(s)}{f^2(s)} ds \right\}. \quad (2.5)$$

We can put $g'(\rho) = 2 \left(\frac{p-1}{p}\right)^{\frac{1}{p}} f(\rho) r(\rho)$, where

$$r(\rho) := \frac{F(\rho)^{-\frac{1}{p}}}{f(0)} - \int_0^{\rho} (F(\rho) - F(s))^{-\frac{1}{p}} \frac{f'(s)}{f^2(s)} ds. \quad (2.6)$$

Then we have the following result.

Lemma 2.5. *We have*

- (1) *if $\lim_{s \rightarrow \infty} \frac{f(s)}{s^{p-1}} = 0$, then $\lim_{s \rightarrow +\infty} g(s) = +\infty$, and*
- (2) *if $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = +\infty$, then $\lim_{s \rightarrow +\infty} g(s) = 0$.*

Proof. See [7, Theorem 7]. \square

3. Main results

In this section we give our main results.

Let $\alpha > 0$, $\rho > 0$ and $m \in (0, \rho/\alpha)$. Then, from (2.2) and (2.3), we have

$$\lambda^{\frac{1}{p}} = \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \left(\int_0^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} + \int_{\alpha m}^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} \right). \quad (3.1)$$

Using symmetry and Eq. (2.1) we deduce the following equation:

$$\left(\frac{p-1}{p} \right) m^p = \lambda (F(\rho) - F(\alpha m)). \quad (3.2)$$

Then

$$\lambda^{\frac{1}{p}} = \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \frac{m}{(F(\rho) - F(\alpha m))^{\frac{1}{p}}}. \quad (3.3)$$

From (3.1) and (3.3) we obtain

$$\int_0^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} + \int_{\alpha m}^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} = \frac{m}{(F(\rho) - F(\alpha m))^{\frac{1}{p}}}. \quad (3.4)$$

We prove the following result.

Theorem 3.1. *Let $\alpha > 0$ and $\rho > 0$. Then there exists a unique $m = m^*(\alpha, \rho) \in (0, \frac{\rho}{\alpha})$, such that (3.4) is satisfied for $m = m^*$. Further m^* is continuously differentiable. Also there is a unique $\lambda = \lambda(\alpha, m^*(\alpha, \rho))$ given by either (3.1) or (3.3) for which (1.1) has a unique solution (u, λ) , with $\|u\| = \rho$, $u'(1) = -m^*(\alpha, \rho)$ and*

$$x_0 = \left(\frac{p-1}{p} \right)^{\frac{1}{p}} \lambda^{\frac{1}{p}} \int_0^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}}.$$

Proof. Let $G : (0, \frac{\rho}{\alpha}) \rightarrow \mathbb{R}_+$ be defined by

$$G(m) := \int_0^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} + \int_{\alpha m}^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} \quad (3.5)$$

and $H : (0, \frac{\rho}{\alpha}) \rightarrow \mathbb{R}_+$ be defined by

$$H(m) := \frac{m}{(F(\rho) - F(\alpha m))^{\frac{1}{p}}}. \quad (3.6)$$

From (3.5) and (3.6) we have

$$G'(m) = -\frac{\alpha}{(F(\rho) - F(\alpha m))^{\frac{1}{p}}} < 0 \quad (3.7)$$

and

$$H'(m) = \frac{p + \alpha m f(\alpha m)(F(\rho) - F(\alpha m))^{-1}}{p(F(\rho) - F(\alpha m))^{\frac{1}{p}}} > 0. \quad (3.8)$$

Then G is decreasing from $2 \int_0^\rho \frac{dw}{(F(\rho) - F(w))^{1/p}}$ to $\int_0^\rho \frac{dw}{(F(\rho) - F(w))^{1/p}}$, and H is increasing from 0 to $+\infty$. Hence there exists a unique $m = m^*(\alpha, \rho)$ such that $G(m^*) = H(m^*)$.

The regularity of m^* is deduced from the implicit function theorem. \square

From the previous theorem we deduce the following result.

Corollary 3.2. *Let $\alpha > 0$ and $\rho > 0$. Then the bifurcation diagram (λ, ρ) of positive solutions of (1.1) is described by*

$$\lambda^{\frac{1}{p}}(\alpha, \rho) = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}} + \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_{\alpha m^*(\alpha, \rho)}^\rho \frac{dw}{(F(\rho) - F(w))^{\frac{1}{p}}}. \quad (3.9)$$

We now investigate the number of positive solutions for $\alpha = 0$.

Theorem 3.3. *Let $\alpha = 0$. If $f' \leq 0$ and $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = 0$, then the problem (1.1) has exactly one positive solution for all $\lambda > 0$.*

Proof. From (2.6) we have $r > 0$. Then g is strictly increasing. This implies that g is invertible. Hence $\rho = g^{-1}(\lambda^{1/p})$. Moreover, using Lemma 2.5, we get $\lim_{s \rightarrow +\infty} g(s) = +\infty$. \square

Theorem 3.4. *Let $\alpha = 0$. If $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = +\infty$, then there exists $\lambda^* > 0$ such that the problem (1.1) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, and zero positive solution for $\lambda > \lambda^*$.*

Proof. The results come from Lemmas 2.3 and 2.5.

In fact, we have $\lim_{s \rightarrow 0+} g(s) = \lim_{s \rightarrow +\infty} g(s) = 0$. Then g is bounded and reaches its maximum at some value $\rho_0 > 0$. Moreover, $\lambda^* = (g(\rho_0))^p$. \square

In the following theorem we determine the exact number of solutions of (1.1) for $\alpha = 0$.

Theorem 3.5. *Let $\alpha = 0$. If $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = +\infty$, $f'(0) \geq 0$ and $(\frac{f'}{f^3})' \geq 0$, then there exists $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$ the problem (1.1) has exactly two positive solutions, for $\lambda = \lambda^*$ there is one positive solution, and there is no positive solution for $\lambda > \lambda^*$, with $\lambda^* = (\sup\{g(s); s \in (0, +\infty)\})^p$.*

Proof. We have from (2.6)

$$r'(\rho) = -\frac{f(\rho)}{p} \frac{F(\rho)^{\frac{1-p}{p}}}{f(0)} - f(\rho) \left\{ \frac{f'(0)}{f(0)^3} F(\rho)^{-\frac{1}{p}} + \int_0^\rho (F(\rho) - F(s))^{-\frac{1}{p}} \left(\frac{f'(s)}{f^3(s)} \right)' ds \right\}. \quad (3.10)$$

From the hypothesis of the theorem we have, in one hand, $r' < 0$ which implies that r is strictly decreasing, and in the other hand, $\lim_{s \rightarrow 0^+} g(s) = \lim_{s \rightarrow +\infty} g(s) = 0$ which implies that g admits at least one maximum at $\rho_0 > 0$.

We have $g'(\rho_0) = 0$, then $r(\rho_0) = 0$. Hence $r(\rho) < 0$ for $\rho > \rho_0$ and $r(\rho) > 0$ for $\rho < \rho_0$. This implies that g is strictly increasing for $\rho < \rho_0$, and strictly decreasing for $\rho > \rho_0$. \square

Now we prove the existence of multiple solutions for $\alpha > 0$.

Theorem 3.6. *Let $\alpha > 0$. Then*

- (1) *if $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = 0$, the problem (1.1) has at least one positive solution for all $\lambda > 0$, and*
- (2) *if $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = +\infty$, there exists $\lambda_0^* = (\sup\{h_{m^*(\alpha,s)}(s); s \in (0, +\infty)\})^p$ such that (1.1) has at least two positive solutions for $\lambda \in (0, \lambda_0^*)$, and no positive solution for $\lambda > \lambda_0^*$.*

Proof. We have $g(\rho) \leq 2h_{m^*}(\rho) \leq 2g(\rho)$, for all $\rho > 0$. From Theorems 3.3 and 3.4, we deduce the results of the theorem. \square

4. Concluding remarks

In this work we have studied a boundary value problem of the one-dimensional p -Laplacian with positive parameters α and λ . We have proved the existence of multiple positive solutions using the quadrature method. Our results combine and generalize the works of [4,7].

The quadrature method can be used only for autonomous ordinary and partial differential equations, in the non-autonomous case other methods could be used (bifurcation, degree theory, numerical methods, etc.).

Very interesting concrete models involving the p -Laplacian arising from population biology are developed in [22], on which we are working in order to obtain a general results.

Finally, note that interests of recent works are focused on the weak solutions [19,23].

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